HOSSAM GHANEM

(22) 3.* Tangent Line; Vertical Tangent Line And Corner(B)

Example 1	60 *October* 31 , 2011 (4 points) Let $f(x) = x^{7/3} - 7x^{1/3}$. Find all the point on the graph of f at which (a) the tangent line is horizontal (b) the tangent line is vertical .

Solution

$$f(x) = x^{7/3} - 7x^{1/3}$$

$$f^\backslash(x) = \frac{7}{3}x^{\frac{4}{3}} - \frac{7}{3}x^{-\frac{2}{3}} = \frac{7}{3}x^{-\frac{2}{3}}(x^2-1) = \frac{7(x^2-1)}{3x^{\frac{2}{3}}}$$

H.T at $x^2 - 1 = 0 \quad \rightarrow \quad x = \pm 1$

V.T at $x^{\frac{2}{3}} = 0 \quad \rightarrow \quad x = 0$

Example 2	51 November 24, 2008
	Show that $f(x) = |\sin x|,$ has a corner in $(-\pi, \pi)$

Solution

$$f(x) = \begin{cases} \sin x & \text{If } 0 < x < \pi \\ 0 & \text{If } x = 0 \\ -\sin x & \text{If } -\pi < x < 0 \end{cases}$$

$$f^\backslash(0^-) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{-\sin x - 0}{x - 0} = \lim_{x\to 0^-} \frac{-\sin x}{x} = -1$$

$$f^\backslash(0^+) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{\sin x - 0}{x - 0} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

$\therefore \ f^\backslash(0^-) \neq f^\backslash(0^+)$

$\therefore f$ has a corner at $x = 0$, *in* $(-\pi, \pi)$

Example 3	52 April 9, 2009 A
	Show that the graph of $f(x) = \dfrac{x^{\frac{2}{3}}}{x-1}$ has a vertical tangent line

Solution

$$f(x) = \frac{x^{\frac{2}{3}}}{x-1} \quad , \quad x \neq 1$$

$$f^\backslash(x) = \frac{(x-1)\cdot\frac{2}{3}x^{\frac{-1}{3}} - x^{\frac{2}{3}}}{(x-1)^2} = \frac{x^{\frac{-1}{3}}\left(\frac{2}{3}(x-1) - x\right)}{(x-1)^2} = \frac{\frac{2}{3}(x-1) - x}{x^{\frac{1}{3}}(x-1)^2}$$

$$\lim_{x\to 0} f^\backslash(x) = \infty$$

$\because \ f$ cont. at $x = 0$

$\therefore f$ has a vertical tangent line at $x = 0$

Example 4

23 April 27,2000

Let $f(x) = \frac{x}{x+1}$ Find all x at which the tangent line to the graph of f is Parallel to the line $4y - x - 3 = 0$

Solution

$L: \quad 4y - x - 3 = 0 \qquad m = \frac{1}{4}$

$f:$

$$f(x) = \frac{x}{x+1}$$

$$f^{\backslash}(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$\therefore \frac{1}{(x+1)^2} = \frac{1}{4}$$

$$(x+1)^2 = 4$$

$$x + 1 = \pm 2$$

$x + 1 = 2$ or $x + 1 = -2$

$\therefore \ x = 1$ or $x = -3$

Example 5

31 June 5, 2008

Find equations of the lines of slope -4 that are tangents to the curve $y = \frac{1}{x}$.

Solution

$$y = \frac{1}{x}$$

$$y^{\backslash} = \frac{-1}{x^2}$$

$$\therefore \frac{-1}{x^2} = -4$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$

$\therefore x = -\frac{1}{2}$ or $x = \frac{1}{2}$

$\therefore y = -2$ or $y = 2$

$\therefore \ m = -4 \qquad p_1\left(-\frac{1}{2}, -2\right) \qquad p_2\left(\frac{1}{2}, 2\right)$

$$y - y_1 = m(x - x_1)$$

$L_1: \quad y + 2 = -4\left(x + \frac{1}{2}\right)$

$$y + 4x + 4 = 0$$

$L_2: \quad y - 2 = -4\left(x - \frac{1}{2}\right)$

$$y + 4x - 4 = 0$$

Example 6

34 June 21, 2009

Show that the curves $f(x) = x^2$ and $g(x) = -x^2 + 4x - 2$ have the same tangent line at their point of intersection

Solution

Intersection point at

$f(x) = g(x)$

$\therefore \; x^2 = -x^2 + 4x - 2$

$\therefore \; 2x^2 - 4x + 2 = 0$

$\therefore \; x^2 - 2x + 1 = 0$

$\therefore \; (x-1)^2 = 0$

$\therefore \; x = 1$

$f^{\backslash}(x) = 2x \quad \rightarrow \quad f^{\backslash}(1) = 2$

$g^{\backslash}(x) = -2x + 4 \quad \rightarrow \quad g^{\backslash}(1) = 2$

$f^{\backslash}(1) = g^{\backslash}(1)$

$\therefore$ the curves have the same tangent line at $x = 1$

Example 7

07/12/2011

(4 points) : Find an equation for the tangent line to the curve $y = 2 + \sin(xy)$ at $x = 0$

Solution

$y = 2 + \sin(xy)$

$y|_{x=0} = 2 + \sin(0) = 2$

$y^{\backslash} = \cos(xy)\,(y + xy^{\backslash})$

$y^{\backslash} = \cos(0)\,(2 + 0) = 2$

$\therefore p(0, 2) \qquad m = 2$

$y - y_1 = m(x - x_1)$

$y - 2 = 2x$

Homework

1	47 November 10, 2007 A Let $f(x) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}} - 2$ Find the point on the graph of f at which the tangent line is vertical
2	32 August 02, 2008 Find the $x-$coordinate of the point where the tangent line to the curve $y = (2x-1)^{\frac{1}{3}} + x^2 + 7$ is vertical.
3	Show that f has vertical tangent at $x = 0$ $f(x) = (x+3)\sqrt[3]{x}$
4	21 May 27. 2001 Let $f(x) = 2 + \sqrt[3]{x^2 - 1}$. Show that the graph of f has a vertical tangent at the point (1 , 2).
5	28 January 13. 2007 Let $f(x) = \frac{3}{8}(8 - x^2)x^{\frac{2}{3}}$. Find the $x-$coordinate of the point at which the tangent line to the graph of f is horizontal and the $x-$coordinate of the point at which the tangent line to the graph of f is vertical.
6	29 June 4. 2007 Let $f(x) = x^{\frac{1}{3}}(x^2 - 3)^{\frac{1}{3}}$. Find the $x-$coordinate of the point at which the tangent line to the graph of f is horizontal and the $x-$coordinate of the point at which the tangent line to the graph of f is vertical.
7	50 November 17, 2008 A Let $f(x) = x^{\frac{5}{3}} - 5x^{\frac{2}{3}} + 1$ Find the x-coordinates of the points at which the graph of f has (a) a horizontal tangent line (b) a vertical tangent line.
8	Find the point on the graph of f which the slop of the tangent line is 3 where $f(x) = 2x - \frac{1}{x}$
9	6 April 8, 1993 Find the values of a , b and c so that the graph of the equation $y = ax^2 + bx + c$, passes through the origin and point (1 , 1) and its tangent line has slop 3 at the point (1 , 1)

Homework

10	Let $f(x) = a\,x^2 - 12x + 8$. Find all values of a such that the tangent Line to the graph of f at $x = 3$ is parallel to the line $y - 6x + 1 = 0$
11	Let $f(x) = 2x\sin x + x + 1$ show that there is a point P on the graph of f at which the tangent line is parallel to the straight line $y - 2x + 1 = 0$
12	Prove that the line tangent to the curve $y = x + 2x^2 - 4x^4$ at the point $(-1\,,\ 0)$ is also tangent to the curve at the point $(\,1\,,\ 2)$
13	Show that f has a corner at $x = 0$; $f(x) = \begin{cases} 2x & \text{if} \quad x \le 0 \\ x^2 & \text{if} \quad x > 0 \end{cases}$
14	Given $f(x) = x^2 + x\cos x - 1$ Use the intermediate value theorem to show that there is a real number c between $-\frac{\pi}{2}$ and 0 such that $f^{\backslash}(c) = 0$
15	Let $f(x) = x^3 + x^2 - x$ Use the intermediate value theorem to show that there is a point on the graph of f at which the tangent line is horizontal
16	58 7April 2011 [3 pts.] At what points on the curve $y = \dfrac{x-1}{x+1}$ is the tangent line parallel to the line $x - 2y = 2$?
17	37 June 6, 2010 Let $f(x) = \dfrac{(x-3)^{\frac{2}{3}}}{x-1}$ (a) Find the x −coordinate(s), if any , of the point(s) on the curve $y = f(x)$ where the tangent line is vertical (b) Find the x −coordinate(s), if any , of the point(s) on the curve $y = f(x)$ where the tangent line is horizontal
18	Show that f has a corner at $x = 2$; $f(x) = \lvert x - 2\rvert + 5$
19	35 August 15, 2009 Find equations of the lines passing through the origin and tangent to the curve $y = x^2 + 1$

18

Show that f has a corner at $x = 2$; $f(x) = |x - 2| + 5$

Solution

$$f(x) = \begin{cases} x - 2 + 5 & \text{If } x > 2 \\ 5 & \text{If } x = 2 \\ -x + 2 + 5 & \text{If } x < 2 \end{cases} \qquad f(x) = \begin{cases} x + 3 & \text{If } x > 2 \\ 5 & \text{If } x = 2 \\ -x + 7 & \text{If } x < 2 \end{cases}$$

$$f^\backslash(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{x + 3 - 5}{x - 2} = \lim_{x \to 2^+} \frac{x - 2}{x - 2} = 1$$

$$f^\backslash(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{-x + 7 - 5}{x - 2} = \lim_{x \to 2^-} \frac{-x + 2}{x - 2} = -1$$

$\therefore \ f^\backslash(2^+) \neq f^\backslash(2^-)$

$\therefore$ f has a corner at $x = 2$

19 35 August 15, 2009

Find equations of the lines passing through the origin and tangent to the curve

$$y = x^2 + 1$$

Solution

$y = x^2 + 1$

$y^\backslash = 2x$

Let the point of tengency $(a, a^2 + 1)$

$\therefore m = 2a$

$\therefore y - y_1 = m(x - x_1)$

$\therefore y - a^2 - 1 = 2a\,(x - a)$

$\therefore$ the line is passing through the point $(0,0)$

$-a^2 - 1 = 2a\,(-a)$

$-a^2 - 1 = -2a^2$

$a^2 = 1$

$a = \pm 1$

at $a = 1$

$\therefore y - 1 - 1 = 2\,(x - 1)$

$y - 2 = 2x - 2$

$y = 2x$

at $a = -1$

$\therefore y - 1 - 1 = -2(x + 1)$

$y - 2 = -2x - 2$

$y = -2x$